Fully Anisotropic String Cosmologies, Maxwell Fields and Primordial Shear

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We present a class of *exact* cosmological solutions of the low energy string effective action in the presence of a homogeneous magnetic fields. We discuss the physical properties of the obtained (fully anisotropic) cosmologies paying particular attention to their vacuum limit and to the possible isotropization mechanisms. We argue that quadratic curvature corrections are able to isotropize fully anisotropic solutions whose scale factors describe accelerated expansion. Moreover, the degree of isotropization grows with the duration of the string phase. We follow the fate of the shear parameter in a decelerated phase where, dilaton, magnetic fields and radiation fluid are simultaneously present. In the absence of any magnetic field a *long string phase* immediately followed by radiation is able to erase large anisotropies. Conversely, if a *short string phase* is followed by a long dilaton dominated phase the anisotropies can be present, in principle, also at later times. The presence of magnetic seeds after the end of the string phase can induce further anisotropies which can be studied within the formalism reported in this paper.

I. FORMULATION OF THE PROBLEM

Suppose that at some time t_1 the Universe becomes transparent to radiation and suppose that, at the same time, the four-dimensional background geometry (which we assume, for simplicity, spatially flat) has a very tiny amount of anisotropy in the scale factors associated with different spatial directions, namely

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)[dy^2 + dz^2], \tag{1.1}$$

where $b(t)$, as it will be clear in a moment, has to be only slightly different from $a(t)$. The electromagnetic radiation propagating along the x and y axes will have a different temperature, namely [1,2]

$$T_x(t) = T_1 \frac{a_1}{a} = T_1 e^{-\int H(t)dt}, \quad T_y(t) = T_1 \frac{b_1}{b} = T_1 e^{-\int F(t)dt}, \tag{1.2}$$

where $H(t)$ and $F(t)$ are the two Hubble factors associated, respectively with $a(t)$ and $b(t)$. The temperature anisotropy associated with the electromagnetic radiation propagating along two directions with different expansion rates can be roughly estimated, in the limit $H(t) - F(t) \ll 1$, as

$$\frac{\Delta T}{T} \sim \int [H(t) - F(t)]dt = \frac{1}{2}\int \epsilon(t) d\log t \tag{1.3}$$

where, in the second equality, we assumed that the deviations from the radiation dominated expansion can be written as $F(t) \sim (1 - \epsilon(t))/2t$ with $|\epsilon(t)| \ll 1$. As we will see in the following ϵ can be connected with the shear parameter.

The dynamical origin of the primordial anisotropy in the expansion [2] can be connected with the existence of a primordial magnetic field (not dynamically generated but postulated as an initial condition) or with some other sources of anisotropic pressure and, therefore, the possible bounds on the temperature anisotropy can be translated into bounds on the time evolution of the anisotropic scale factors.

In the context of string cosmology [4] there are no reasons why the Universe should not be mildly anisotropic as it was recently pointed out [5]. Indeed, the Universe *can be* anisotropic as a result of the Kasner-like nature of the pre-big-bang solutions. Moreover, we could argue, that the Universe *has to be only mildly anisotropic*. By mildly anisotropic we mean that the scale factors should be *both expanding* (in the String frame). The low energy beta functions can be solved for a metric (1.1) with the result [5]

$$a(t) = \left[-\frac{t}{t_1}\right]^{\alpha}, \quad b(t) = \left[-\frac{t}{t_1}\right]^{\beta}, \quad \phi(t) = (\alpha + 2\beta - 1)\log\left[-\frac{t}{t_1}\right], \tag{1.4}$$

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where $\alpha^2 + 2\beta^2 = 1$. If we want $\dot{a} > 0$ and $\dot{b} > 0$ (expanding solutions) and $\ddot{a} > 0$, $\ddot{b} > 0$ (inflationary solutions) the exponents α and β have to lie in the third quadrant along the arc of the ellipse reported in Fig. 1.

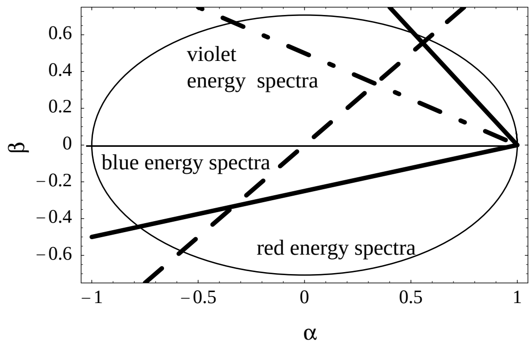

FIG. 1. The axionic logarithmic energy spectra (in frequency) are reported for different dilaton-driven models with Kasner-like exponents (α, β) in the string frame. The models belonging to the third quadrant and localized on the arc of the "vacuum" ellipse $\alpha^2 + 2\beta^2 = 1$ correspond to *anisotropic models with expanding and accelerated scale factors*. If α and β lie on the ellipse but in the first quadrant we have solutions of the low energy beta functions which are both contracting (in the String frame). If α and β on the ellipse but either in the second or in the fourth quadrant we have solutions of the low energy beta functions where one of the two scale factors expands and the other contracts. Notice that the dashed (thick) line does correspond to the case of *fully isotropic* solutions (i.e. $\alpha = \beta = -1/\sqrt{3}$) whose intersection with the vacuum ellipse lies in the region of red spectra. *Above* the dot-dashed (tick) line the dilaton *decreases* for $t < 0$ whereas *below* the dot-dashed line the dilaton *increases*. We would be tempted to speculate that to have an increasing dilaton is a *sufficient* condition in order to have a pre-big-bang dynamics. This is *not* the case. Indeed, an *increasing* dilaton is also compatible with α and β in the second quadrant where the scale factors are *not both expanding*. Therefore, an increasing dilaton is not a sufficient condition for anisotropic pre-big-bang dynamics but *only a necessary* condition.

The requirement of having, in the same class of fully anisotropic models, axion spectra decreasing at large distances (i.e. blue frequency spectra) will select a slice exactly in the same region of parameter space [5]. Therefore, general considerations seem to point towards *mildly anisotropic pre-big-bang models*. Since for phenomenological considerations [6] we would like to have blue (or flat) logarithmic energy spectra it seems reasonable to analyze fully anisotropic string cosmological models. Recently it was pointed out that the collapse of a stochastic collection of dilatonic waves (in the Einstein frame) does also lead to an anisotropic pre-big-bang phase [7].

In the framework of an anisotropic pre-big-bang phase it is plausible, from a theoretical point of view, to investigate the role played by a homogeneous magnetic field which could represent a further source of anisotropy. Thus, the first technical point we want to investigate is the possible generalization of the pre-big-bang solutions to the case of a homogeneous magnetic field. This generalization is not so straightforward since, in the low energy limit, the dilaton field is directly coupled to the kinetic term of the Maxwell field

$$S = -\frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g} e^{-\phi} \left[R + g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + ... \right] \tag{1.5}$$

where $F_{\alpha\beta} = \nabla_{[\alpha} A_{\beta]}$ is the Maxwell field strength and ∇_μ is the covariant derivative with respect to the String frame metric $g_{\mu\nu}$. Notice that $H_{\mu\nu\alpha}$ is the antisymmetric field strength. In Eq. (1.5) the ellipses stand for a possible (non-perturbative) dilaton potential and for the string tension corrections parameterized by $\alpha' = \lambda_s^2$ (notice that λ_s is the string scale). In Eq. (1.5) $F_{\mu\nu}$ can be thought as the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$. One might think that by going to the Einstein frame the dilaton can be decoupled from the kinetic term of the Maxwell fields. This is not the case as we discuss, in greater detail, in Appendix A.

In addressing the possible occurrence of an anisotropic phase in the life of the Universe there is an immediate concern. We want to make sure that the amount of anisotropy encoded in the initial conditions will be eventually washed out by the subsequent evolution. It can be shown, in this respect, that an interesting role can be played by the string tension corrections [5]. In fact, by adding the first α' correction to the tree-level action reported in Eq. (1.5) two interesting things happens. On one hand the curvature invariants get regularized and, on the other hand, an anisotropic solution with expanding scale factors gets isotropized. This statement can be made more precise by looking, simultaneously at Fig. 1 and Fig. 2. In Fig. 2 we plot the shear parameter $r(t) = 3[H(t) - F(t)]/[H(t) + 2F(t)]$ for the particular vacuum solution leading to flat axionic spectra in four anisotropic dimensions, namely the case $\alpha = -7/9$, $\beta = -4/9$. This case does correspond in Fig. 1 to the intersection between the full line and the vacuum ellipse in the third quadrant. We can clearly see that for $t \to -\infty$ the solution is anisotropic since $r(t)$ is of order one. The crucial

question for the purpose of the present paper is by how much the quadratic corrections are able to reduce the degree of anisotropy.

For our future convenience we *define* the degree of isotropization as the logarithm (in then basis) of the absolute value of the shear parameter:

$$I(t) = \log |r(t)|. \tag{1.6}$$

The reason for the absolute value in the last equations is that $r(t)$ can change sign depending upon the relative magnitude of H and F. In Fig. 2 we plot the degree of isotropization (at the left) and the shear parameter itself (at the right). We see that the degree of isotropization is a function *of the duration of the string phase*. Thus if we have a very long string phase we can have $I(t) \lll -7$. On the other hand if the duration of the string phase is not too long, then one can naturally have larger values for $r(t)$. Suppose now that the string phase stops at some time t and it is replaced by a radiation dominated phase. Then by the continuity of the scale factors at the transition time the anisotropy in the expansion is translated in a temperature ansisotropy according to Eq. (1.1).

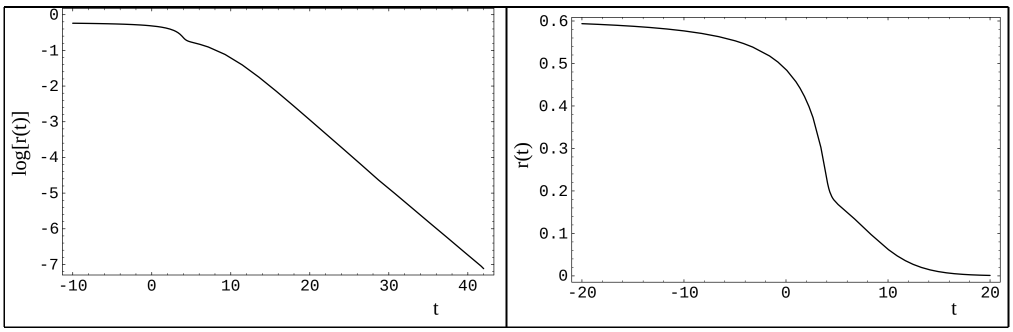

FIG. 2. We plot the evolution of the degree of anisotropy (left) and of the shear parameter itself (right) in the case of a mildly anisotropic solution with $\alpha = -7/9$ and $\beta = -4/9$ corresponding to the case of flat axionic spectra [5] (i.e. the intersection of the thick full line with the vacuum ellipse in the third quadrant of Fig. 1). As we can see for $t \to -\infty$, $r(t) \to [-7/9 + 4/9]/[-7/9 - 4/9] = 3/11$. For $t \to +\infty$ the shear parameter and the degree of anisotropy gets reduced as the result of the addition of of higher order string tension corrections (see also Section IV). We notice that the degree of isotropization crucially depends upon the *duration* of the string phase (left). In a *short* string phase $I(t)$ can be of the order of -3, -4. If the string phase is very long the degree of anisotropy can be much smaller than -7. Notice that, by only looking at the right picture we might guess (in a wrong way) that the shear parameter tends (for $t \to +\infty$) to a constant (small) value. The crucial point we want to stress is that this is not the case as we can understand by looking at $I(t)$ which is *always a decreasing function of the cosmic time*.

Therefore, it could happen, depending upon the duration of the string phase, that either $I(t) \gtrsim 10^{-5}$ or $I(t) \lesssim 10^{-5}$ right after the end of the string phase (under the assumption that radiation sets in instantaneously). Many questions can arise at this point. Will the radiation evolution completely erase the primordial anisotropy? If this is not the case, should we exclude, on a phenomenological ground, too short string phases since they might potentially conflict with the observed value of the large scale anisotropy? Which is the role of a primordial magnetic field in the picture? We want to recall, in fact, that even if we would not include a magnetic field in the pre-big-bang phase such a field will be anyway dynamically generated by parametric amplification of the electromagnetic (vacuum) fluctuations [8] and it will evolve, in the radiation dominated (post-big-bang) phase affecting the evolution of $r(t)$ [1]. Certainly one can argue that the magnetic field generated from the electromagnetic vacuum fluctuations is stochastic in nature as all the other fields (gravitons, scalar fields...) produced as the result of parametric amplification of quantum mechanical initial conditions [9,10]. Therefore, we could conjecture that the obtained magnetic field does not have any preferred direction. We want to point out that this conclusion has been reached by looking at the magnetic field generation in a completely homogeneous and isotropic dilaton-driven phase. If the background *is not completely isotropic* during the dilaton driven evolution, then, we can argue, in analogy with the graviton production in anisotropic backgrounds of Bianchi type I, that the produced background has different properties in different spatial directions [11].

Finally, the considerations and the various questions we just stated hold in the case of transition from the string phase to the radiation dominated phase. However, it could happen that before reaching the radiation phase the background evolves toward a state of decreasing dilaton and then, we should treat again the same issues in the case of a background with decreasing dilaton coupling. These are the problems we want to investigate in the second part of the present paper.

Our plan is the following. In Section II we will discuss the magnetic solutions of the low energy beta functions whereas in Section III we will discuss their physical interpretation. In Section IV we will investigate the role of the α'

corrections and we will try to follow the evolution of the degree of anisotropy all along the string phase. In Section V we will study the fate of the anisotropy in a decelerated phase and we will address the issue of possible bounds on the duration of the string phase from too large temperature anisotropies. Section VI contains our concluding remarks. Some useful technical results concerning the description of anisotropies and of magnetic fields in the String and in the Einstein frames are collected in Appendix A.

II. MAGNETIC SOLUTIONS

Consider a spatially flat background configuration with vanishing antysimmetric field strength ($H_{\mu\nu\alpha} = 0$) and vanishing dilaton potential. The dilaton depends only on time and the metric will be taken fully anisotropic since we want to study possible solutions with a homogeneous magnetic field which is expected to break the isotropy of the background:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \tag{2.1}$$

This choice of the metric corresponds to a synchronous coordinate system in which the time parameter coincides with the usual cosmic time (i.e. $g_{00} = 1$ and $g_{0i} = 0$).

By varying the effective action with respect to ϕ, $g_{\mu\nu}$ and the vector potential A_μ we get, respectively

$$R - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.2}$$

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = \frac{1}{2}\left[\frac{1}{4}\delta_\mu^\nu F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\beta}F^{\nu\beta}\right] - \frac{1}{2}\delta_\mu^\nu g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \nabla_\mu\nabla^\nu\phi + \delta_\mu^\nu\Box\phi, \tag{2.3}$$

$$\nabla_\mu\left[e^{-\phi}F^{\mu\nu}\right] = 0. \tag{2.4}$$

where ∇_μ denotes covariant differentiation with respect to the metric of Eq. (2.1). Inserting Eq. (2.2) into Eq. (2.3) we get that Eq. (2.3) can be expressed as

$$R_\mu^\nu + \nabla_\mu\nabla^\nu\phi + \frac{1}{2}F_{\mu\alpha}F^{\nu\alpha} = 0. \tag{2.5}$$

Consider now a homogeneous magnetic field directed along the x axis. The generalized Maxwell equations (2.4) and the associated Bianchi identities can then be trivially solved by the field strength $F_{yz} = -F_{zy}$. The resulting system of equations (2.2), (2.5) can than be written, in the metric of Eq. (2.1), as

$$2\ddot{\phi} + 2(H + F + G)\dot{\phi} - \dot{\phi}^2 - 2\left[HF + FG + FG + H^2 + F^2 + G^2 + \dot{H} + \dot{F} + \dot{G}\right] + \frac{B^2}{2b^2c^2} = 0, \tag{2.6}$$

$$\ddot{\phi} = H^2 + F^2 + G^2 + \dot{H} + \dot{F} + \dot{G}, \tag{2.7}$$

$$H\dot{\phi} = HF + HG + H^2 + \dot{H}, \tag{2.8}$$

$$F\dot{\phi} = HF + FG + F^2 + \dot{F} - \frac{B^2}{2b^2c^2}, \tag{2.9}$$

$$G\dot{\phi} = HF + FG + G^2 + \dot{G} - \frac{B^2}{2b^2c^2}, \tag{2.10}$$

where Eq. (2.6) is the dialton equation and Eqs. (2.7)–(2.10) do correspond, respectively to the (00) and (ii) components of Eq. (2.5). Notice that B is the magnetic field intensity. The inclusion of a time-dependent B is in principle possible but in order to be consistent with the Bianchi indentities we would need to include also an electric field (not necessarily homogeneous). Notice that in the limit $B \to 0$ we obtain the well known form of the low energy beta functions in a fully anisotropic metric.

In order to give a general solution of the previous system it is convenient to define a generalized "conformal" time namely

$$e^{-\overline{\phi}}d\eta = dt, \qquad \overline{\phi} = \phi - \log\sqrt{-g}, \tag{2.11}$$

where $\overline{\phi}$ is the "shifted" dilaton which is invariant for duality related vacuum solutions. In terms of η Eqs. (2.2)–(2.10) become, respectively,

$$2\phi'' + \phi'^2 - 2\Sigma' - 2\phi'\Sigma + \Lambda = -\frac{B^2}{2}a^2 e^{-2\phi}, \tag{2.12}$$

$$\phi'' + \phi'^2 - 2\Sigma\phi' + \Lambda - \Sigma' = 0, \tag{2.13}$$

$$\mathcal{H}' = 0, \quad \mathcal{F}' = \frac{B^2}{2}a^2 e^{-2\phi}, \quad \mathcal{G}' = \frac{B^2}{2}a^2 e^{-2\phi}, \tag{2.14}$$

where

$$\Sigma = (\mathcal{H} + \mathcal{F} + \mathcal{G}), \quad \Lambda = 2(\mathcal{H}\mathcal{F} + \mathcal{H}\mathcal{G} + \mathcal{F}\mathcal{G}), \tag{2.15}$$

with the obvious notation that $(\log a)' = \mathcal{H}$, $(\log b)' = \mathcal{F}$, $(\log c)' = \mathcal{G}$ (the prime denotes differentiation with respect to η). By subtracting Eq. (2.13) from Eq. (2.12) we obtain

$$\phi'' - \Sigma' = -\frac{B^2}{2}a^2 e^{-2\phi}. \tag{2.16}$$

Using the remaining equations in Eq. (2.16) in order to eliminate Σ' we get a general form of the solution which reads

$$a(\eta) = a_0 e^{\mathcal{H}_0\eta}, \quad b(\eta) = b_0 e^{\mathcal{F}_0\eta} e^{\phi}, \quad c(\eta) = a_0 e^{\mathcal{G}_0\eta} e^{\phi}, \tag{2.17}$$

where ϕ satisfies

$$\phi'' + \phi'^2 - 2\mathcal{H}_0\phi' - \Lambda_0 = 0, \quad \Lambda_0 = 2(\mathcal{H}_0\mathcal{F}_0 + \mathcal{F}_0\mathcal{G}_0 + \mathcal{F}_0\mathcal{G}_0). \tag{2.18}$$

By solving this last equation and by inserting the (trial) solution (2.17) into Eqs. (2.12)–(2.10) we will get a set of consistency relations among the different integration constants which will determine a general form of the solutions in terms of the (undetermined) set of initial conditions.

Let us notice that by repeatedly combining the equations of motion we obtain a very useful relation, namely $2\phi'' = (Ba)^2 \exp[-2\phi]$. Since the left hand side of this last equation is positive definite, we have also to demand, for physical consistency that, from Eq. (2.18), $-\phi'^2 + 2\mathcal{H}_0\phi' + \Lambda_0 > 0$, which also implies

$$\mathcal{H}_0 + \sqrt{\mathcal{H}_0^2 + \Lambda_0} < \phi' < \mathcal{H}_0 - \sqrt{\mathcal{H}_0^2 + \Lambda_0}. \tag{2.19}$$

Notice that this is a requirement to be satisfied in the presence of a constant *magnetic* field and it does change in the presence of a constant *electric* field (indeed for a constant electric field directed along the x direction and in the absence of any associated magnetic field we would have that, in the same conformal time parameterization $\phi'' < 0$).

The wanted solution is

$$a(\eta) = a_0 e^{\mathcal{H}_0\eta}, \quad b(\eta) = b_0 e^{\phi_0} e^{(\mathcal{F}_0 + \mathcal{H}_0 + \Delta_0)\eta}\left[e^{-2\Delta_0\eta} + e^{\phi_1}\right], \quad c(\eta) = c_0 e^{\phi_0} e^{(\mathcal{G}_0 + \mathcal{H}_0 + \Delta_0)\eta}\left[e^{-2\Delta_0\eta} + e^{\phi_1}\right], \tag{2.20}$$

$$\phi(\eta) = \phi_0 + (\mathcal{H}_0 + \Delta_0)\eta + \log\left[e^{\phi_1} + e^{-2\Delta_0\eta}\right], \tag{2.21}$$

where, by consistency with all the other equations we have

$$\Delta_0 \equiv \sqrt{\mathcal{H}_0^2 + \Lambda_0} = \frac{a_0}{2\sqrt{2}} e^{-(\phi_0 + \phi_1)} B. \tag{2.22}$$

A particular solution of the system is fixed by fixing the value of the magnetic field in string units. Before discussing the physical properties of the solution (2.21) we want to see in which way the vacuum solutions appear in the time parameterization defined by Eq. (2.11). The vacuum solutions are quite straightforward using η and they correspond to constant Hubble factors and linear dilaton, namely

$$\mathcal{H} = \mathcal{H}_0, \quad \mathcal{F} = \mathcal{F}_0, \quad \mathcal{G} = \mathcal{G}_0, \quad \phi(\eta) = \phi_0 + \phi_2\eta, \tag{2.23}$$

with the condition,

$$\phi_2 = \mathcal{H}_0 + \mathcal{F}_0 + \mathcal{G}_0 \pm \sqrt{\mathcal{H}_0^2 + \mathcal{F}_0^2 + \mathcal{G}_0^2}. \tag{2.24}$$

It is straightforward to see that these solutions are indeed Kasner-like by working out the cosmic time picture from Eq. (2.11). Indeed, with little effort we can see that

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\alpha_1}, \quad b(t) = b_0\left(\frac{t}{t_0}\right)^{\alpha_2}, \quad c(t) = c_0\left(\frac{t}{t_0}\right)^{\alpha_3}, \tag{2.25}$$

where

$$\alpha_1 = \frac{\mathcal{H}_0}{\mathcal{H}_0 + \mathcal{F}_0 + \mathcal{G}_0 - \phi_2}, \quad \alpha_2 = \frac{\mathcal{F}_0}{\mathcal{H}_0 + \mathcal{F}_0 + \mathcal{G}_0 - \phi_2}, \quad \alpha_3 = \frac{\mathcal{G}_0}{\mathcal{H}_0 + \mathcal{F}_0 + \mathcal{G}_0 - \phi_2} \tag{2.26}$$

Using Eq. (2.24) we can obtain that the three (cosmic time) exponents satisfy the Kasner-like condition, namely $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and the $\pm$ ambiguity of Eq. (2.24) simply refers to the two (duality related) branches.

Before concluding the present Section we want to mention that the equations of motion can also be integrated in the case where the coupling of the dilaton to the Maxwell fields is slightly different [12] from the one examined in this Section, namely in the case where the action is written as

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}e^{-q\phi}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{2.27}$$

with $q \neq 0$.

In this case the equations of motion in the generalized conformal time η can be written as

$$2\phi'' + {\phi'}^2 - 2\Sigma\phi' + \Lambda - 2\Sigma' = -\frac{q+1}{2}B^2a^2e^{-(q+2)\phi} \tag{2.28}$$

$$\phi'' + {\phi'}^2 - 2\Sigma\phi' - \Sigma' + \lambda = -\frac{q}{4}B^2a^2e^{-(q+2)\phi} \tag{2.29}$$

$$\mathcal{H}' = \frac{q}{4}B^2a^2e^{-(q+2)\phi}, \quad \mathcal{F}' = \frac{q+2}{4}B^2a^2e^{-(q+2)\phi}, \quad \mathcal{G}' = \frac{q+2}{4}B^2a^2e^{-(q+2)\phi}. \tag{2.30}$$

Notice that in this case the system changes qualitatively but it can still be integrated. In fact, by following the procedure outlined in the previous paragraphs we obtain a decoupled equation for ϕ'

$$-\frac{\phi''}{q+1} - \frac{q^2+2q+2}{2(q+1)^2}{\phi'}^2 + \frac{2\mathcal{H}_0}{q+1}\phi' + \Lambda_0 = 0 \tag{2.31}$$

This equation can be easily integrated providing the solutions to the whole system in the case $q \neq 0$.

III. ANALYSIS OF THE MAGNETIC SOLUTIONS

In order to analyze the previous solution let us assume, for sake of simplicity, that $b(\eta) \equiv c(\eta)$. Suppose now, that our initial state is weakly coupled (i.e. ϕ_0 sufficiently negative) and $(\mathcal{H}_0, \mathcal{F}_0)$ are sufficiently small in string units. Even though $\mathcal{H}_0$ and $\mathcal{F}_0$ are of the same order they do have different initial conditions. Let us therefore analyze what is the role of a magnetic field, small in string units. The discussion becomes simpler if we start looking at the η behavior of the solutions. From Eqs. (2.21) we can argue that the evolution of $\phi'(\eta)$ is quite straightforward since for $\eta \to -\infty$ and $\phi' \to \mathcal{H}_0 - \Delta_0$ whereas for $\eta \to +\infty$ we have that $\phi'(\eta) \to \mathcal{H}_0 + \Delta_0$. Moreover, as we discussed in the previous Section, $\phi'' > 0$ which implies that ϕ' is always an increasing function of η. As a consequence we will also have that $\mathcal{F}(\eta)$ will also be an increasing function of η whereas $\mathcal{H}$ is frozen to its constant value given by $\mathcal{H}_0$. According to Eq. (2.22) the magnetic field intensity in string units determines, together with ϕ_0 the value of Δ_0. Therefore, we conclude that the magnetic field intensity is essentially responsible of the amount of growth of the dilaton energy since it turns out that $|\phi'(+\infty) - \phi'(-\infty)| = 2\Delta_0$. Thus, by increasing the magnetic field we observe an increase of the dilaton energy density. Notice, however, that this growth cannot be so large. Indeed, from Eq. (2.22) we notice that the relation between Δ_0 and B is suppressed by the second power of the (initial) coupling constant $g(\phi) = \exp(\phi/2)$. So, with our initial conditions, Δ_0 is *small from the beginning* since ϕ_0 is very negative. Notice that to have small coupling is essentially dictated by the form of our original action which cannot be trusted if $g(\phi)$ gets of order one. This qualitative behavior of the solutions is illustrated in Fig. 3–4 in few specific cases.

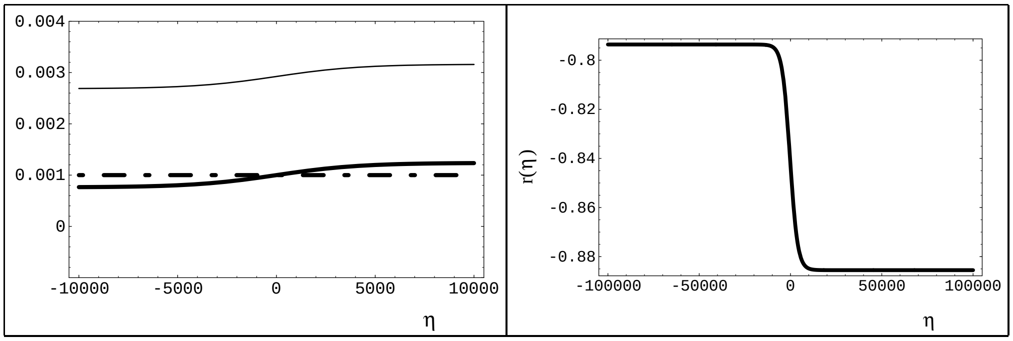

FIG. 3. We plot the magnetic solutions given in Eqs. (2.21) in the case where $\mathcal{H}_0 = 0.001$, $B = 01$, $\phi_0 = -5$. In the *left* plot we illustrate the behavior of $\mathcal{H}$ (*dot-dashed line*), $\mathcal{F}$ (*thin line*) and ϕ' (*full thick line*). At the right we illustrate the behavior of the shear parameter. We notice that, in spite of the appearance, the shear parameter does not decrease, asymptotically but it tends to a constant value of order one, in sharp contrast with what happens if we include quadratic corrections to the tree-level action (see next Section). We see that $\mathcal{H}$, $\mathcal{F}$ and ϕ' are *almost* constant if the magnetic field is *small* in String units.

As a a reference value for $\mathcal{H}_0$ and for ϕ_0 we take 0.001 and -5 since we want to deal with small curvatures in String units and small couplings. In Fig. 3 (left) we report the solutions in the case of a quite large magnetic field $B \sim 1$. The dot-dashed line denotes the behavior of $\mathcal{H}$ (constant), the full (thin) line denotes the evolution of $\mathcal{F}(\eta)$ whereas the behavior of the full (thick) line denotes the evolution of ϕ'.

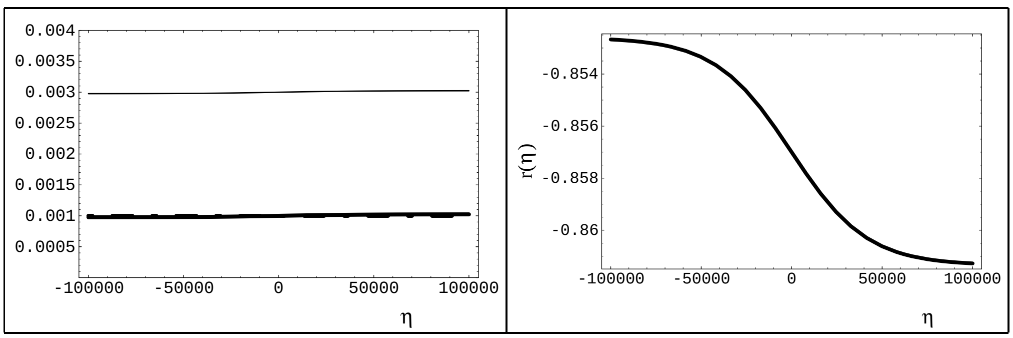

FIG. 4. We plot the same solution given in Fig. 3 but with a different value of the magnetic field which is now $B = 0.001$. As in the previous plot at the *left* we report the evolution of $\mathcal{H}$ (*dot-dashed line*), $\mathcal{F}$ (*thin line*)and ϕ' (*full thick line*). Notice that in this case the growth of ϕ' is so minute that the dot-dashed line and the full thick lines are almost indistinguishable. This is due to the fact that ϕ' continuously interpolates between $\mathcal{H}_0 - \Delta_0$ and $\mathcal{H}_0 + \Delta_0$ and $\Delta_0 \sim 0.001e^{-5}$. At the *right* we report the evolution of the shear parameter for the same solution.

It is interesting to compare this plot with the ones illustrated in Fig. 4 where (left plot) we report the solution (2.21) with the very same choice of initial conditions of Fig. (3) but with different magnetic field, namely $B = 0.001$ in String units. We see that the growth of ϕ' is reduced (by two order of magnitude).

What about the evolution of the shear parameter? Let us look at the left plots reported in Fig. 3, 4 where the evolution of

$$r(\eta) = \frac{3[\mathcal{H}(\eta) - \mathcal{F}(\eta)]}{[\mathcal{H}(\eta) + 2\mathcal{F}(\eta)]} \tag{3.1}$$

is given for the same set of parameters we just discussed. In Fig. 3 (where $B \sim 0.1$) we would say that the shear parameter decreases. This is certainly true but if we look at the numbers we can see that $r(\eta)$ starts of order one and essentially remains of order one. In Fig. 4 (i.e., $B = 0.01$) $r(\eta)$ seem to decrease, but again, a more correct statement would be that $r(\eta)$ remains of order one (i.e. $r(+\infty) \sim -0.88$), up to a transition period. Thus, in spite of its smallness, the magnetic field has always the property of *conserving* the anisotropy. The negative sign in $r(\eta)$ is only due to the fact that $\mathcal{F} > \mathcal{H}$ for any η. Thus, as far as the tree-level evolution is concerned we can say that no isotropization is observed as a consequence of the inclusion of the magnetic field. This situation should be contrasted with what happens when the string tension corrections are included.

The second effect we see is that the smaller is the value of the magnetic field, the milder is the transition to the vacuum solutions. We said, in the previous section that the vacuum solutions are essentially given by $\mathcal{H} =$ constant,

$\mathcal{F} = constant$, $\phi' =$ constant. For $\eta \to +\infty$ the magnetic solutions should then tend to the vacuum solutions. Thus, the influence of the magnetic field is mainly in the transition period from the initial state to the (asymptotic vacuum) solution.

In order to show that the large η behavior of our solutions is Kasner-like, let us look explicitly to the cosmic time description of the system. Let us look at the evolution of our system in cosmic time. The relation between η and t can be obtained by inserting Eqs. (2.21) into Eq. (2.11) whose integrated version becomes

$$\frac{t}{a_0 b_0^2 e^{\phi_0}} = e^{\phi_1} \frac{e^{2(\mathcal{F}_0+\mathcal{H}_0)\eta + \Delta_0 \eta}}{[2(\mathcal{F}_0 + \mathcal{H}_0) + \Delta_0]} + \frac{e^{2(\mathcal{F}_0+\mathcal{H}_0)\eta - \Delta_0 \eta}}{[2(\mathcal{F}_0 + \mathcal{H}_0) - \Delta_0]}. \tag{3.2}$$

Notice that the only two independent quantities determining the relation between η and t are $\mathcal{H}_0$ and Δ_0 (which is related to B according to Eq. (2.22)) since $\mathcal{F}_0$ can be expressed in terms of Δ_0 and $\mathcal{H}_0$.

If we now take the large η behavior of Eq. (3.2) and we substitute back into Eqs. (2.21) we do find that the two scale factors can be expressed, in cosmic time, as

$$a(t) \sim t^{\alpha}, \quad b(t) \sim t^{\beta}, \tag{3.3}$$

where

$$\alpha = \frac{\mathcal{H}_0}{2(\mathcal{H}_0 + \mathcal{F}_0) + \Delta_0}, \quad \beta = \frac{\mathcal{F}_0 + \mathcal{H}_0 + \Delta_0}{2(\mathcal{H}_0 + \mathcal{F}_0) + \Delta_0}. \tag{3.4}$$

We can notice that $\alpha^2 + 2\beta^2 = 1$. This can be easily seen by recalling the definition of $\Delta_0^2 = \mathcal{H}_0^2 + 2(2\mathcal{H}_0\mathcal{F}_0 + \mathcal{F}_0^2)$ which can be deduced from Eqs. (2.22) and (2.18).

So, close to the singularity, for $\eta \to \infty$ the solutions are Kasner-like. Notice that the presence of the vector field (with the specific four-dimensional coupling required by the low energy effective action) does not lead to any type of oscillatory stage. This is evolution is in sharp contrast with one obtains in the case of a vector field (included following the Kaluza-Klein ansatz) in the Einstein frame [13]. The reason for this difference stems from from the difference in the couplings. In Fig. 5 the evolution of $\dot{\phi}$, $H(t)$ and $F(t)$ are illustrated in the case of $B \sim 0.1$. Notice that $t \to 0$ does correspond to the large η behavior. The singularity is located, in this example for $t \sim 2$.

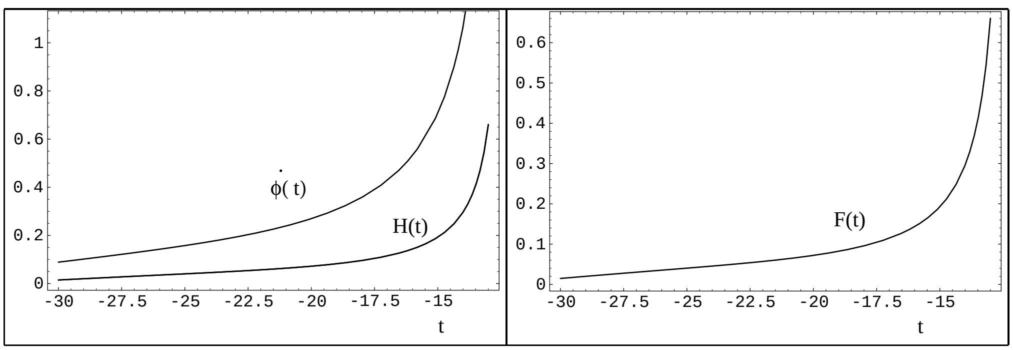

FIG. 5. We illustrate the behavior of $H(t)$, $\dot{\phi}$ (left plot) and of $F(t)$ (right plot) in the case of a magnetic field of the order of $B \sim 0.1$ in String units.

In Fig. 6 we report the same solution illustrated in Fig. 5 but with a smaller magnetic field, $B = 0.01$. As one can expect the singularity and the vacuum limit is reached before than in the previous case.

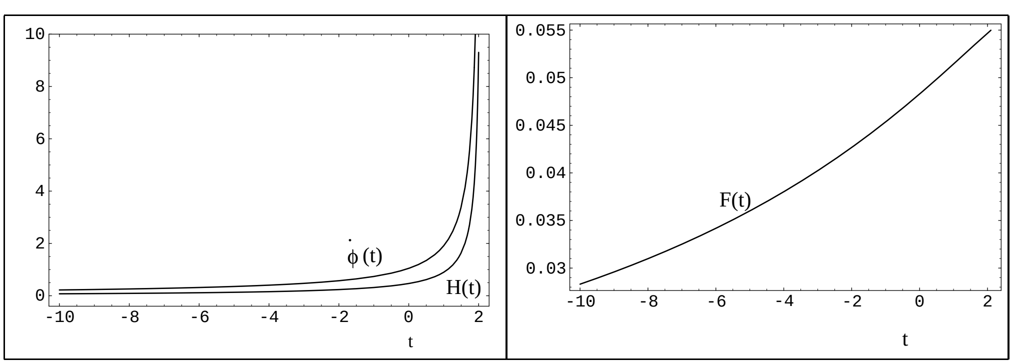

FIG. 6. We illustrate the behavior of $H(t)$, $\dot{\phi}$ (left plot) and of $F(t)$ (right plot) in the case of a magnetic field of the order of $B \sim 0.1$ in String units.

IV. MAGNETIC SOLUTIONS AND THE HIGH CURVATURE REGIME

In the previous two Sections we examined fully anisotropic (magnetic) cosmologies using the tree level action. In this Section we are going to extend and complement our results with the addition to the tree-level action of the first string tension correction. In the presence of the first α' correction the action becomes

$$S = -\frac{1}{2\lambda_s^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{\omega\lambda_s^2}{4}\left(R_{GB}^2 - (g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)^2\right)\right], \tag{4.1}$$

where R_{GB}^2 is the Gauss-Bonnet invariant expressed in terms of the Riemann, Ricci and scalar curvature invariants

$$R_{GB}^2 = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{4.2}$$

and ω is a numerical constant of order 1 which can be precisely computed depending upon the specific theory we deal with (for instance $\omega = -1/8$ for heterotic strings). In principle the magnetic field should also appear in the corrections to the tree-level action. However, it turns out that, the terms which would be significant for the type of configurations examined in this paper, the gauge fields appear beyond the first α' correction. For example terms like $F^{\alpha\beta}F^{\mu\nu}R_{\alpha\beta\mu\nu}$ will appear to higher order in α'. Moreover, possible terms like $F^{\alpha\beta}R_{\alpha\beta}$ are vanishing in the case of constant and homogeneous magnetic field directed, say, along the x direction. Finally, other possibly relevant terms (involving contractions of the vector potentials with the Ricci or Riemann tensors) are forbidden by gauge invariance.

Of course, the spirit of our analysis of the high curvature regime is only semi-quantitative. After all, the first curvature correction can be thought as illustrative, since, ultimately *all* the α' corrections will turn out to be important. The only way of turning-off the possible effect of curvature corrections is to have *explicit* solutions whose curvature invariants are already regularized at tree level like in the examples presented in [14] where it was shown that there exist weakly inhomogeneous solutions of the tree level action which are regular and geodesically complete without the addition of any curvature or loop corrections to the tree level action. If this is not the case, the curvature corrections have to be certainly included. The inclusion of higher order curvature corrections can be studied either in the Einstein frame [15] or in the String frame [16]. Another interesting approach has been outlined in [17].

For a reasonably swift derivation of the equations of motion it is useful to write the action in terms of the relevant degrees of freedom. In order to get correctly the constraint equation (i.e. the (00) component of the beta functions) it is appropriate to keep the lapse function $N(t)$ it its general form

$$g_{\mu\nu} = \mathrm{diag}[N(t)^2, -e^{2\alpha(t)}, -e^{2\beta(t)}, -e^{2\beta(t)}], \tag{4.3}$$

where we parameterized the two scale factors with an exponential notation. Only *after* the equations of motion have been derived we will set $N(t) = 1$ corresponding to the synchronous time gauge. In the metric (4.3) the previous action (4.1) becomes, after integration by parts,

$$S = \frac{1}{2\lambda_s^2}\int dte^{\alpha+2\beta-\phi}\left\{\frac{1}{N}\left[-\dot{\phi}^2 - 2\dot{\beta}^2 - 4\dot{\alpha}\dot{\beta} + 2\dot{\alpha}\dot{\phi} + 4\dot{\beta}\dot{\phi} - \frac{1}{2}B^2e^{-4\beta(t)}\right] + \frac{\omega\lambda_s^2}{4N^3}\left[8\dot{\phi}\ \dot{\alpha}\ \dot{\beta}^2 - \dot{\phi}^4\right]\right\}. \tag{4.4}$$

By varying Eq. (4.4) with respect to the lapse function $N(t)$ and imposing, afterwards, the cosmic time gauge we get the constraint

$$\dot{\phi}^2 + 2\dot{\beta}^2 + 4\dot{\alpha}\dot{\beta} - 2\dot{\alpha}\dot{\phi} - 4\dot{\beta}\dot{\phi} - \frac{1}{2}B^2e^{-4\beta} + \left[\frac{3}{4}\dot{\phi}^4 - 6\dot{\alpha}\dot{\phi}\dot{\beta}^2\right] = 0, \tag{4.5}$$

where we took string units $\lambda_s = 1$ and $\omega = 1$. By varying the action with respect to α, β and ϕ we get, for $N(t) = 1$ the diagonal components of the beta functions and the dilaton equation,

$$4\ddot{\beta} - 2\ddot{\phi} + (4\dot{\beta} - 2\dot{\phi})(\dot{\alpha} + 2\dot{\beta} - \dot{\phi}) - 2\left[(\dot{\alpha} + 2\dot{\beta} - \dot{\phi})\dot{\phi}\dot{\beta}^2 + \ddot{\phi}\dot{\beta}^2 + 2\dot{\beta}\ddot{\beta}\dot{\phi}\right] + L(t) = 0, \tag{4.6}$$

$$4(\ddot{\beta} + \ddot{\alpha} - \ddot{\phi}) + 2B^2e^{-4\beta} + 4(\dot{\alpha} + \dot{\beta} - \dot{\phi})(\dot{\alpha} + 2\dot{\beta} - \dot{\phi}) - 4\left[\ddot{\beta}\dot{\alpha}\dot{\phi} + \dot{\beta}\ddot{\alpha}\dot{\phi} + \dot{\beta}\dot{\alpha}\ddot{\phi} + \dot{\beta}\dot{\alpha}\dot{\phi}(\dot{\alpha} + 2\dot{\beta} - \dot{\phi})\right] + 2L(t) = 0, \tag{4.7}$$

$$2(\ddot{\phi} - \ddot{\alpha} - 2\ddot{\beta}) + 2(\dot{\phi} - \dot{\alpha} - 2\dot{\beta})(\dot{\alpha} + 2\dot{\beta} - \dot{\phi}) - \left[2\dot{\beta}(\ddot{\alpha}\dot{\beta} + 2\dot{\alpha}\ddot{\beta}) + (2\dot{\alpha}\dot{\beta}^2 - \dot{\phi}^3)(\dot{\alpha} + 2\dot{\beta} - \dot{\phi}) - 3\dot{\phi}^2\ddot{\phi}\right] - L(t) = 0, \tag{4.8}$$

where we defined

$$L(t) = -\dot{\phi}^2 - 2\dot{\beta}^2 - 4\dot{\alpha}\dot{\beta} + 2\dot{\alpha}\dot{\phi} + 4\dot{\beta}\dot{\phi} - \frac{1}{2}B^2 e^{-4\beta} + \frac{1}{4}(8\dot{\phi}\dot{\alpha}\dot{\beta}^2 - \dot{\phi}^4). \tag{4.9}$$

We can numerically integrate this system. The technique is very similar to the one used in the case of fully anisotropic string cosmologies with zero magnetic field [5,16]. Our results are illustrated in Fig. 8 and 9. There are two physically different cases.

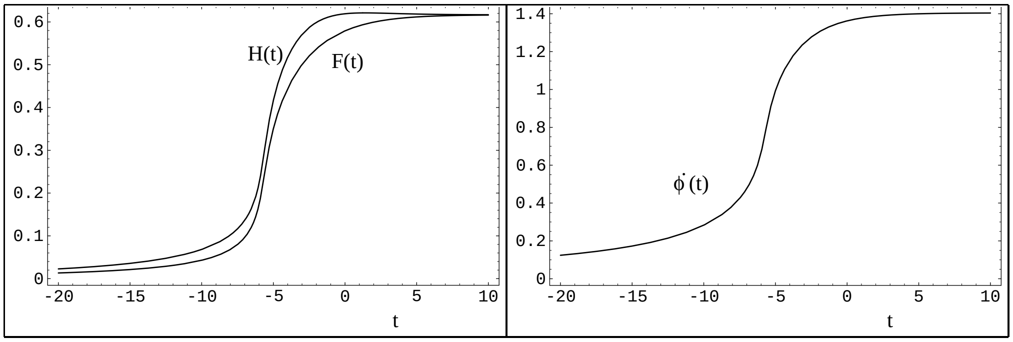

FIG. 7. We report the result of the numerical integration of Eqs. (4.8) in the case where the initial conditions are mildly anisotropic. We choose $B = 0.01$.

We integrate this system by imposing, as initial conditions small curvature and small dilaton coupling. We also impose mildly anisotropic initial conditions (corresponding to $\dot{a} > 0$, $\dot{b} > 0$ and $\ddot{a} > 0$, $\ddot{b} > 0$). One of the results stressed in the previous Section is that the tree-level solutions with magnetic field are singular. Moreover, close to the singularity no oscillatory behavior is present. Therefore, as time goes by and as the solutions approach the singularity the vacuum solutions are recovered. One can then argue that if the α' corrections become relevant when the vacuum regime has been already recovered, nothing should change with respect to the case where $B = 0$. This in some sense is what happens, but not exactly.

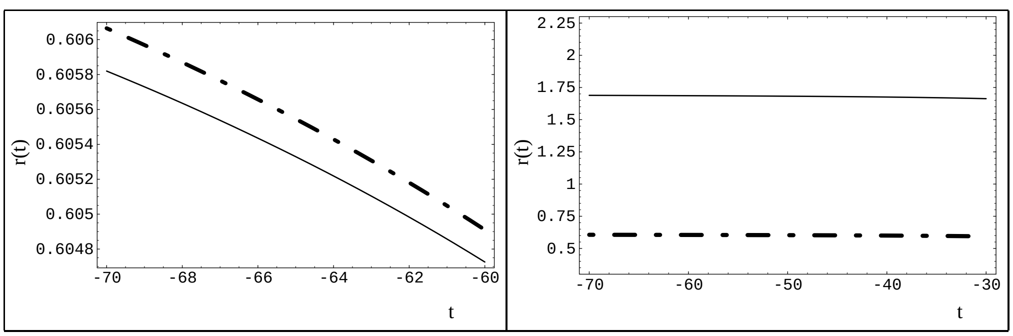

FIG. 8. We illustrate the numerical solutions of the equations of motions *with* string tension corrections and we plot the shear parameter. We select expanding initial conditions (in the String frame). In this particular example (left plot) we choose the magnetic field to be $B = 0$ (dot dashed line), and $B = 0.01$ (full line). The difference between the two cases is quite minute and in order to show it we do not plot the full evolution of $r(t)$ (which goes to zero for $t > 0$). At the right we choose $B = 0.01$ and we change the initial condition within the vacuum solutions. The dot dashed line is for $b(t) \sim t^{-4/9}$ and the full line for $b(t) \sim t^{-0.2}$. Also in this second case the difference is quite minute. We decided to plot only a limited range of time steps in order to stress the numerical difference between different values of the initial shear and of the primordial magnetic field. The full picture looks like the ones reported in Fig. 9.

If the magnetic field is large in string units (i.e. $B > 1$) this system evolves towards a singularity, as expected. In this case, it is certainly true that the tree level solutions evolve towards their vacuum limit. However, in this limit the solutions get more and more anisotropic. So the hypothesis $B > 1$ simply contradicts the assumption of mild anisotropy. From our point of view it is more interesting the case with $B < 1$. In this case mildly anisotropic initial conditions will be attracted towards an isotropic fixed point. One can understand this looking at Eqs. (4.8). For $t \to 0$ $b(t)$ is monotonically increasing (if we select, as we do, expanding initial conditions). Now the magnetic field is always suppressed by $b^{-4}(t) = e^{-4\beta(t)}$ terms in Eqs. (4.8). Therefore, an initially small magnetic field will become more and

more sub-leading. The solutions will then be attracted towards the quasi-isotropic fixed point $\dot{\alpha} \sim \dot{\beta} \sim 0.606$ and $\dot{\phi} \sim 1.414$.

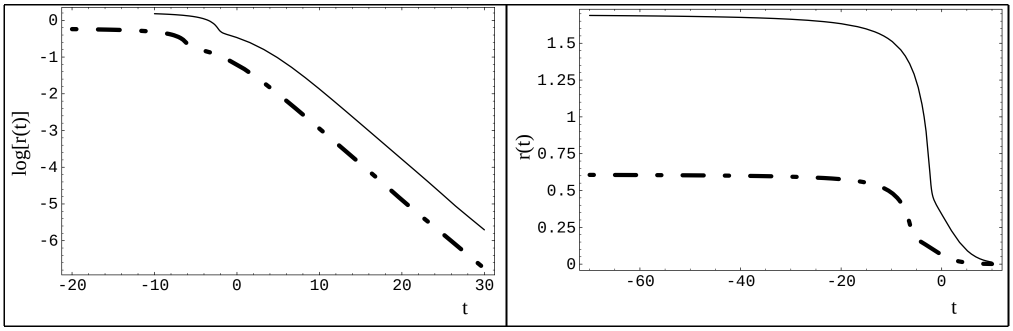

FIG. 9. We report the evolution of the shear parameter in the String phase. With the dot-dashed line we have the case $B = 0$, with the full line the case $B = 0.01$. If a magnetic field is present the shear parameter is larger than in the case of zero magnetic field. It is of crucial importance to notice, for our purposes, that the shear parameter does not tend towards a small (finite value) but it decreases towards 0. In order to stress this aspect we also plot the logarithm (in ten basis) of the modulus of the shear parameter. We can see, as previously discussed, that the duration of the string phase is proportional to the drop in $r(t)$.

In this case the curvature invariants are finite everywhere. We say that the fixed point is quasi-isotropic since the amount of anisotropy depends upon the duration of the high curvature scale. For example, as illustrated in Fig. 9, it can well happen that for a very short stringy phase the amount of anisotropy measured by the shear parameter $r(t)$ will be of the order of 10^{-3}.

V. THE FATE OF THE ANISOTROPIES IN THE POST-BIG-BANG EVOLUTION

In this Section we would like to investigate the fate of the anisotropies possibly present in the dilaton-driven and in the String phase. In particular we would like to understand in which limit these anisotropies can be either reduced to an acceptable value or completely washed out. For numerical purposes we find useful to exploit, in the present Section, the Einstein frame picture. The evolution equations in the Einstein frame are obtained in Appendix A. By linearly combining Eqs. (A.8), (A.9) and (A.10) of Appendix A we obtain a more readable form of the system, namely

$$\dot{H} + H(H+2F) = \frac{\rho}{6} - \frac{w}{2}e^{-\phi}, \tag{5.1}$$

$$\dot{F} + F(H+2F) = \frac{\rho}{6} + \frac{w}{2}e^{-\phi}, \tag{5.2}$$

$$\ddot{\phi} + (H+2F)\dot{\phi} = we^{-\phi}, \tag{5.3}$$

$$(H+2F)^2 - (H^2+2F^2) = \frac{\dot{\phi}^2}{2} + we^{-\phi} + \rho, \tag{5.4}$$

$$\dot{\rho} + \frac{4}{3}(H+2F)\rho = 0, \quad \dot{w} + 4Fw = 0, \quad w = \frac{B^2}{2b^4}, \tag{5.5}$$

where we assumed the presence of a radiation fluid $p = \rho/3$ accounting for the field modes excited, via gravitational instability, during the dilaton driven phase [18] and re-entering in the post-big-bang phase. It is in fact well known that the ultraviolet modes of a field parametrically amplified behave as a radiation fluid [19].

From Eqs. (5.1)–(5.5) we can directly obtain the evolution equations for the quantities we are interested in, namely

$$r(t) = \frac{3(H-F)}{H+2F}, \quad n(t) = \frac{H+2F}{3}, \quad q(t) = \frac{w(t)}{\rho(t)}, \tag{5.6}$$

where $n(t)$ is the mean expansion parameter $r(t)$ is the shear anisotropy parameter and $q(t)$ the fraction of magnetic energy in units of radiation energy. Notice that $q(t)$ *does not* correspond to the critical fraction of magnetic energy density since, in principle, we have to take into account the energy density of the dilaton field. By subtracting Eq. (5.2) from Eq. (5.1) and by using the constraint (5.4) together with the definitions (5.6) we get the shear evolution. By

summing Eqs. (5.1) and (5.2) with similar manipulations we get the evolution of n. The resulting system, equivalent to the one of Eqs. (5.1)–(5.5), but directly expressed in terms of n, r and q reads

$$\dot{n}r + \dot{r}n + 3n^2 r = -q\rho e^{-\phi}, \tag{5.7}$$

$$3\dot{n} + 9n^2 = \frac{\rho}{2}(1 + qe^{-\phi}), \quad \dot{q} - \frac{4}{3}nrq = 0, \tag{5.8}$$

$$6n^2(1 - \frac{r^2}{9}) = \frac{\dot{\phi}^2}{2} + \rho(1 + qe^{-\phi}), \quad \dot{\rho} + 4n\rho = 0, \quad \ddot{\phi} + 3n\dot{\phi} = \rho q e^{-\phi}. \tag{5.9}$$

We want now to study the combined action of the dilaton and of a radiation fluid in a post-big-bang phase. Let us start from the case of medium size anisotropies. Suppose, in other words that after a string phase of intermediate duration the shear parameter is of the order of 10^{-3}. As we saw from the previous Section this value is not excluded. Before analyzing the general case let us recall the results of the case where the dilaton is absent. In this case the system of Eqs. (5.7)–(5.9) can be simplified as follows

$$\dot{n}r + \dot{r}n + 3n^2 r = -q\rho, \tag{5.10}$$

$$3\dot{n} + 9n^2 = \frac{\rho}{2}(1 + q), \quad \dot{q} - \frac{4}{3}nrq = 0, \tag{5.11}$$

$$6n^2(1 - \frac{r^2}{9}) = \rho(1 + qe^{-\phi}), \quad \dot{\rho} + 4n\rho = 0. \tag{5.12}$$

We are interested in the case of magnetic fields which are under-crtical , namely $q(t_0) < 1$ where t_0 is the initial integration time. Suppose, moreover, that $r(t_0) \sim 10^{-3}$. Needless to say that this is just an illustrative value for r and *it is not meant to be of any particular theoretical relevance.* As we stated clearly in the previous Section (see for instance Fig. 8) if the string phase is long the initial shear parameter can be much smaller. The results of the numerical integration are summarized in Fig. 10 where we report the logarithm (in ten basis) of the modulus of the shear parameter.

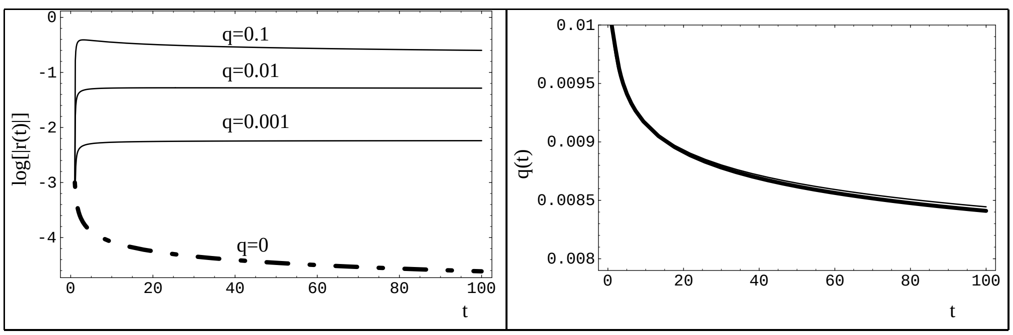

FIG. 10. We report the results of the numerical integration of Eqs. (5.10)–(5.12). We take $r(t_0) \simeq 10^{-3}$ and we integrate the system forward in time for different values of $q(t_0)$. In the left plot we illustrate the evolution of the logarithm (in ten basis) of the modulus of the shear parameter. With the dot-dashed (thick) line we represent the case $q = 0$ where the shear parameter is known to decrease sharply as $t^{-1/2}$. If, initially, the critical balance between the magnetic energy density and the radiation energy density increases, then r gets attracted towards and asymptotic value as described by Eqs. (5.13). At the right (full thick line) we report the evolution of q for the case $q(t_0) = 0.01$. With the thin line we report the qualitative estimate based on Eqs. (5.13) and obtained by solving, approximately, Eqs. (5.10)–(5.12) for $r < 1$ and $q < 1$.

At the bottom of the left plot, with the dot-dashed line, is reported the case of zero magnetic field (i.e. $q(t_0) = 0$). This is the simplest case since we can approximately solve the system of Eqs. (5.10)–(5.12) in the limit of small r. From Eq. (5.10) we can find that a consistent solution is

$$\dot{r} \simeq -\frac{r}{2t} - \frac{3q}{t}, \quad \dot{q} \simeq \frac{2}{3t} rq. \tag{5.13}$$

So, if $q = 0$ we can clearly see that $r(t) \sim t^{-1/2}$. This is nothing but the well known result that in a radiation dominated phase the shear parameter decreases as $1/\sqrt{t}$ and it is precisely what we find, numerically, in the dot-dashed line of Fig. 10 (left plot). If we switch-on the magnetic field we also know, from Eqs. (5.13) that the anisotropy will not decrease forever but it will reach an asymptotic value which crucially depends upon the balance between the

magnetic energy density and the radiation energy density. In fact from eqs. (5.13) we see that $\dot{r} = 0$ for $|r(t)| \to 6|q(t)|$ and this is precisely what we observe in the full lines of Fig. 10 where the integration of the Eqs. (5.10)–(5.12) is reported for different (initial) values of q. We see that the asymptotic value of attraction for r is roughly six times the initial value of q. The evolution of $q(t)$ when $\dot{r}(t) \to 0$ can be simply obtained by integrating once the second of Eqs. (5.13) for the case $r \sim -6q$. The result is that, taking for instance $q(t_0) = 0.01$, $q(t) \sim 1/\{4\log[(t/t_0)] + 100\}$. This simple curve is reported in Fig. 10 at the right (full thin line). As we can see the full thick line (result of the numerical integrations is practically indistinguishable).

The conclusion we draw for our specific case is very simple. If the string phase is not too long and if sizable anisotropies (i.e. $r \sim 10^{-3}$) are still present at the end of the stringy phase, then , in the approximation of a sudden "freeze-out" of the dilaton coupling, the left over of the primordial anisotropy will be washed out *provided* the magnetic field is completely absent. If a tiny magnetic seed directed along the same direction of the anisotropy is present, then in spite of the primordial anisotropy, the shear parameter will be attracted to a constant value essentially fixed by the size of the seed.

Let us now investigate the opposite case. Let us imagine of switching-off completely the radiation background. Then the only non trivial variables determining the shear parameter and the mean expansion will be the dilaton coupling and the magnetic energy density. In this case Eqs. (5.7)–(5.9) do simplify as follows

$$\dot{n}r + \dot{r}n + 3n^2 r = -we^{-\phi}, \tag{5.14}$$

$$3\dot{n} + 9n^2 = \frac{w}{2}e^{-\phi}, \quad \dot{w} + 4(n - \frac{nr}{3})w = 0, \tag{5.15}$$

$$6n^2(1 - \frac{r^2}{9}) = \frac{\dot{\phi}^2}{2} + we^{-\phi}, \quad \ddot{\phi} + 3n\dot{\phi} = \rho q e^{-\phi}. \tag{5.16}$$

As in the previous case let us assume, in order to fix our ideas and in order to have a swift comparison, that $r(t_0) \sim 10^{-3}$. The results of our analysis are reported in Fig. 11.

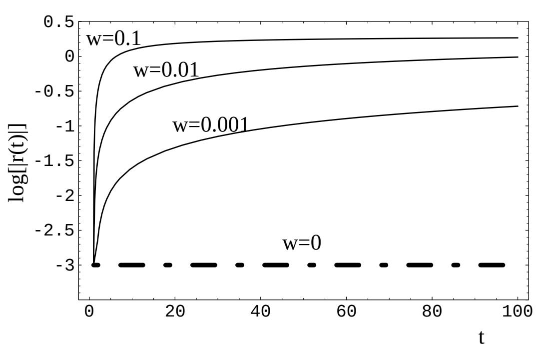

FIG. 11. We report the resluts of the numerical integration of Eqs. (5.14)–(5.16) for different initial values of the magnetic energy density in Planck units. As we can see (dot-dashed line) the shear parameter remains fixed to its initial value if the magnetic field is completely absent. As soon as the magnetic field increases the shear parameter gets attracted towards a fixed point whose typical anisotropy is of order one. As in the previous cases we took $r(t_0) \sim 10^{-3}$ and we also took $\phi(t_0) \sim -0.1$.

Suppose first of all that the magnetic field is switched-off, then, as we see from the dot-dashed line the anisotropy is completely conserved. The introduction of a tiny magnetic seed will eventually make the situation even worse in the sense that the anisotropy will eventually grow to a constant value which depends on the magnetic field intensity. In Fig. 11 we see (full lines) that already with a magnetic energy density of the order of 10^{-3} (in Planck units) the increase in the anisotropy cannot be neglected. Therefore, if the transition from the string phase to the radiation dominated phase does occur through an intermediate dilaton dominated phase of decreasing coupling we have to accept that the anisotropy will be conserved *provided* the magnetic field is absent. If a magnetic field is present the anisotropy gets gets frozen to a constant value which can be non negligible depending upon the size of the magnetic seed. In very rough terms our analysis seems to disfavor mildly anisotropic models with a short string phase and with the simultaneous occurrence of a long dilaton dominated phase prior to the onset of the radiation epoch. So the combination of *short* string phase, *long* dilaton driven phase and radiation phase *shorter than usual* might give too large shear, say, prior to nucleosynthesis. Our conclusion can be even more problematic if it is present a sizable magnetic field.

With the knowledge coming from the two previous examples we can easily analyze the general case given in Eqs. (5.7)–(5.9). The dynamics of the system will depend upon the different balance in the initial conditions. Let us suppose, firstly that the dilaton kinetic energy is slightly smaller (by two orders of magnitude) than the radiation energy density. The results of the numerical integration of the shear parameter are reported in Fig. 12.

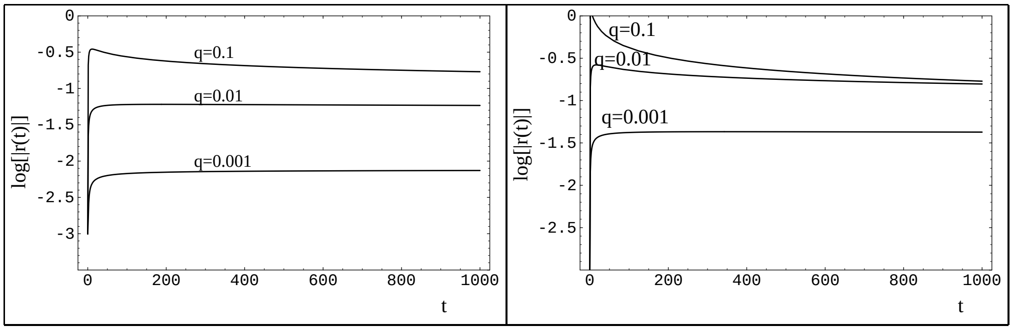

FIG. 12. In this two plots we report the results of the integration of Eqs. (5.7)–(5.9) in the general case where magnetic field, dilaton and radiation fluid are present simultaneously. In these particular plot we assume that, initially, the dilaton energy density is smaller (by two orders of magnitude) than the radiation energy density. The left plot corresponds to the case of $\phi(t_0) = -0.2$ and the right plot corresponds to the case of $\phi(t_0) = -2$. Different initial q are reported in both cases. As in the previous cases we assumed $r(t_0) \sim 10^{-3}$.

We can clearly see that by increasing the magnetic field the anisotropy will increase. The dilaton evolution clearly affects this process as we can argue by comparing Fig. 12 with Fig. 10 where the dialton was absent. It is also interesting to notice that the initial value of the coupling can be of some relevance. In Fig. 12 we reported the result of the integration for two values of the initial coupling, namely $\phi_0 = -0.1$ (at the left) and $\phi_0 = -2$ (at the right). The effect can be explained from Eqs. (5.7)–(5.9) where we can see that by tuning the initial coupling to smaller values we amplify the effect of the magnetic energy density which appears always as $qe^{-\phi}$.

If the radiation energy density is sub-leading with respect to the dilaton kinetic energy we can expect, on the basis of the intuition developed in the previous cases that the shear parameter will stay basically constant if the magnetic field is turned off and it will increase to a constant value. This is more or less what happens. In Fig. 13 we report the results of a numerical integration of the shear parameter in the case where the radiation energy is hundred times smaller than the dilaton kinetic energy.

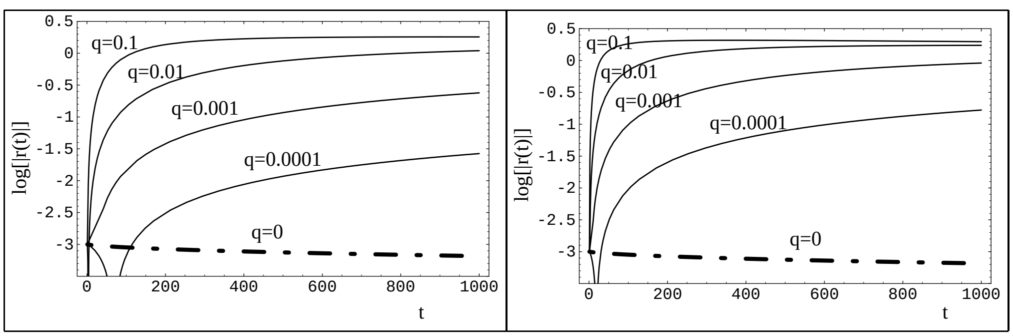

FIG. 13. We report the numerical integration of the system given in Eqs. (5.7)–(5.9) for another set of parameters. In this case we assume that the radiation energy density is hundred times smaller than the dilaton energy density and we watch the relaxation of the anisotropy for different values of the initial magnetic fields. As in the previous case the left picture refers to the case where $\phi(t_0) \sim -0.2$ whereas the right plot refers to the case where $\phi(t_0) \sim -2$. These plots have to be compared with the ones reported in Fig. 10 and Fig. 11. We see that, unlike in the case of Fig. 10, the integration with $q(t_0) = 0$ does not lead to an exactly constant anisotropy. For larger values of $q(t_0)$ the shear parameter saturates, as expected.

The left picture refers to the case of $\phi(t_0) \sim -0.2$ and the right figure refers to the case of $\phi(t_0) \sim -2$. In both cases we can notice that in the limit of zero magnetic field the anisotropy is not *exactly* constant as in the case of $\rho = 0$ reported in Fig. 11 but it has some mild slope and it decreases. At the same time, in contrast with Fig. 10 the asymptotic value of the anisotropy is not exactly the one we could guess on the basis of the argument illustrated in Fig. 10.

We want to stress, finally an important point. Suppose that the dilaton kinetic energy and the radiation energy are both present but the magnetic field is zero ($q(t_0) = 0$). Then we said that the anisotropy has a mild slope. The question is how mild. This point is addressed in Fig. 14 where the results of a numerical integration are reported for fixed dilaton kinetic energy and for radiation energy ten, hundred, thousand ad ten thousand smaller than the dilaton kinetic energy.

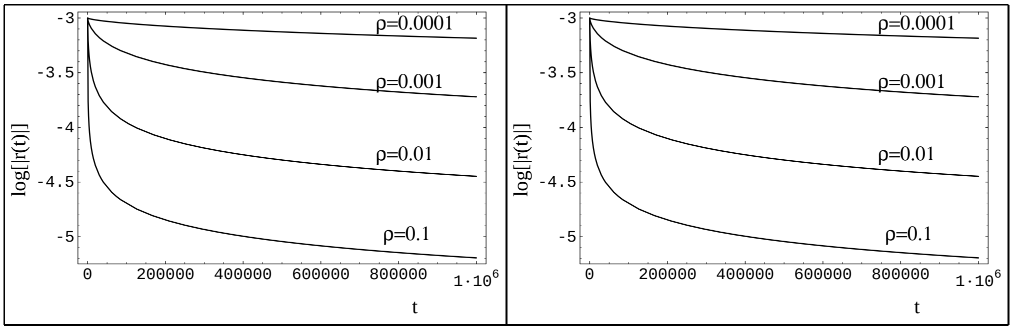

FIG. 14. In this plots we assume that the magnetic field is initially zero. We also assume that the radiation energy density is smaller than the dilaton energy density which is assumed to be of order one. The plot at the left is for $\phi(t_0) = -2$ whereas the plot at the right is for the case $\phi(t_0) = -10$. Nothing changes in the two plots since, for both cases $q = 0$. We notice that the behavior of the anisotropy is qualitatively similar both to the behaviors reported in Fig. 10 and Fig. 9 but with crucial differences. For instance, take the case with $\rho \sim 0.1$ at the left. In this case the radiation and dilaton energy densities are almost comparable. So there is some functional similarity with the plot of Fig. 9. Look anyway at the time scale and take into account that the initial value of the anisotropy is the same in both cases. The same reduction of the shear parameter obtained in 100 time steps in the case of pure radiation is now obtained in 10^6 time steps. So the dilaton slows down the decay of the anisotropy in the absence of magnetic field. If the radiation energy density is gradually removed from the system the anisotropy is (almost) constant.

We see that, in the best case, the anisotropy decrease of two orders of magnitude in 10^6 time steps. This behavior should be compared with the sudden decrease experienced by the anisotropy in the case of Fig. 10 (dot-dashed line) where the dilaton was absent. Therefore, the presence of the dilaton in a radiation background slows down considerably, in the absence of magnetic field, the sudden fall-off of the shear parameter.

The conclusion we can draw from the analysis of the general case is that, also in the presence of the dilaton field, the magnetic field has the effect of increasing the anisotropy. Moreover, if the dilaton is larger than (or comparable with) the radiation fluid at the moment of the end of the string phase the anisotropy does decrease not so easily.

VI. CONCLUDING REMARKS

In this paper generalized the solutions of the low energy beta functions to the case where a magnetic field is present. We found that these solutions can be expressed analytically in the String frame. We investigated their limit in the vicinity of the singularity and we noticed that they approach, monotonically, the well known Kasner-like vacuum solutions.

We addressed a very simple question which can be phrased in the following way. Suppose that the dilaton-driven evolution of the string cosmological models is fully anisotropic thanks, for instance, to a primordial magnetic seed. The obvious issue is to understand how (and possibly when) these fully anisotropic solutions reach the complete isotropic stage. By complete isotropic stage we meant, more quantitatively, a value of the shear parameter smaller, say, than 10^{-7}. We reached a number of conclusions.

For physical reasons we want to deal, from the very beginning, with anisotropic solutions whose scale factors are both expanding and accelerated. In this situation it is not forbidden to have large anisotropies and shear parameters of order one prior to the onset of the string phase.

Now, if the *string phase is sufficiently long* and if it is *immediately followed by a radiation dominated phase*, then, any pre-existing shear coming from the initial condition is very efficiently washed out to arbitrary small values depending upon the duration of the string phase. Of course, this conclusion holds *provided* no magnetic field is present in the ordinary decelerated phase. If a magnetic field is present, then the amount of shear appearing in our present Universe will not be determined by the primordial shear but by the shear *induced by the magnetic seed itself*. The role of the magnetic field might also be of some help but only in the case where the primordial shear is really large *after* the string phase. In this case a very small magnetic field can effectively attract the large anisotropy towards a smaller value.

We can also have the opposite situation. Namely we can have the case where the *string phase is very short* and it is *followed by a phase dominated by the dilaton* until sufficiently small scales lasting from the end of the sting phase until the onset of radiation which should anyway occur *before nucleosynthesis*. In this second scenario the shear will not be erased for two reasons. First of all because the duration of the string phase is proportional to the shear

reduction. Secondly, because in a dilaton dominated phase (in the absence of magnetic field) the primordial shear is conserved.

APPENDIX A: FROM THE STRING TO THE EINSTEIN FRAME

In this Appendix we show explicitly how to get the Einstein frame action for our system. The transformation from the String to the Einstein frame *does not* remove the coupling of the dilaton to the kinetic term of the gauge fields. The String frame action is simply

$$S = -\int d^4x\sqrt{-g}e^{-\phi}\biggl[R + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\biggr]. \tag{A.1}$$

In four dimensions the transformation from the String frame metric ($g_{\mu\nu}$) to the Einstein frame metric ($G_{\mu\nu}$) simply reads

$$g_{\mu\nu} = e^{\phi}G_{\mu\nu}, \quad \sqrt{-g} = e^{2\phi}\sqrt{-G}. \tag{A.2}$$

Therefore,

$$R = e^{-\phi}\biggl[\mathcal{R} - 3G^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - \frac{3}{2}G^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\biggr], \tag{A.3}$$

$$-\sqrt{-g}e^{-\phi}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\sigma\beta} = -\sqrt{-G}e^{-\phi}F_{\alpha\beta}F_{\rho\sigma}G^{\alpha\rho}G^{\sigma\beta}, \tag{A.4}$$

the derivatives in Eq. (A.3) are covariant with respect to the metric $G_{\mu\nu}$. Thus, the Einstein frame action can be written as

$$S_E = \int d^4x\sqrt{-G}\biggl[-\mathcal{R} + \frac{1}{2}G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{4}e^{-\phi}F_{\alpha\beta}F^{\alpha\beta}\biggr]. \tag{A.5}$$

Again from this equation we can derive the Equations of motion. We will just report the essential points. The Equations of motion derived from the action of Eq. (A.5) read

$$\begin{aligned}
&\mathcal{R}_\mu^\nu - \frac{1}{2}\delta_\mu^\nu\mathcal{R} = \frac{1}{2}\biggl[T_\mu^\nu(d) + e^{-\phi}T_\mu^\nu(m) + T_\mu^\nu(r)\biggr],\\
&G^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = \frac{1}{4}e^{-\phi}F_{\alpha\beta}F^{\alpha\beta}, \qquad \nabla_\alpha\biggl[e^{-\phi}F^{\alpha\beta}\biggr] = 0
\end{aligned} \tag{A.6}$$

with

$$T_\mu^\nu(d) = \partial_\mu\phi\partial^\nu\phi + \frac{1}{2}\delta_\mu^\nu G^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \quad T_\mu^\nu(m) = -F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{4}\delta_\mu^\nu F_{\alpha\beta}F^{\alpha\beta}, \quad T_\mu^\nu(f) = \mathrm{diag}(\rho, -p, -p, -p), \tag{A.7}$$

where on top of the energy momentum tensors of the dilaton (i.e. $T_\mu^\nu(d)$) and of the Maxwell fields (i.e. $T_\mu^\nu(m)$) we also added the energy momentum tensor of the fluid sources (i.e. $T_\mu^\nu(f)$) because of the considerations reported in Section V. In the fully anisotropic metric given in Eq. (2.1) the previous equations of motion become

$$F^2 + 2HF = \frac{\dot\phi^2}{4} + \frac{B^2}{4b^4}e^{-\phi} + \frac{\rho}{2}, \tag{A.8}$$

$$3F^2 + 2\dot F = -\frac{1}{4}\dot\phi^2 + \frac{B^2}{4b^4}e^{-\phi} - \frac{p}{3}, \tag{A.9}$$

$$\dot H + \dot F + H^2 + F^2 + HF = -\frac{\dot\phi^2}{4} - \frac{p}{2} - \frac{B^2}{4b^4}e^{-\phi}, \tag{A.10}$$

$$\ddot\phi + (H + 2F)\dot\phi = \frac{e^{-\phi}}{2}\frac{B^2}{b^4}, \tag{A.11}$$

$$\dot\rho + (H + 2F)(\rho + p) = 0. \tag{A.12}$$

Concerning these equations two technical comments are in order. First of all the over-dot denotes the derivation with respect to the *cosmic time of the Einstein frame* which is related to the cosmic time of the String frame (used, for instance in Section II) as

$$dt_E = e^{-\phi/2}dt_s, \quad a_E(t_E) = e^{-\frac{\phi}{2}}a_s(t_s), \quad b_E(t_E) = e^{-\frac{\phi}{2}}b_s(t_s). \tag{A.13}$$

In the same way $H_E = (\log a)^{\cdot}$ and $F_E = (\log b)^{\cdot}$ are the Hubble factors defined in the Einstein frame. We will not denote explicitly this distinction but we would like to remind that the various quantities defined in Sections II, II, and IV are defined in the String frame whereas the discussion reported in Section V refers to the Einstein frame.

Most of the times one can check that all the "physical" quantities are invariant with respect to a change of frame. Typical examples of this property are the spectra of the fields excited by the dilaton growth in the case of the Kasner-like "vacuum" solutions derived in Section II. Suppose in fact to compute the axionic (or gravitonic or dilatonic) spectra in the String frame. Then Suppose to redo the same exercise in the Einstein frame with the transformed solutions. The two spectra will be equal. It is curious to notice that the shear parameter is only approximately invariant under conformal rescaling. In order to show this aspect let us consider the dilaton-driven (vacuum) solutions reported in Eq. (1.4) and written in the String frame

$$a_s(t_s) = \left[-\frac{t_s}{t_1}\right]^{\alpha}, \quad b(t_s) = \left[-\frac{t_s}{t_1}\right]^{\beta}, \quad \phi(t_s) = (\alpha + 2\beta - 1)\log\left[-\frac{t_s}{t_1}\right], \tag{A.14}$$

with $\alpha^2 + 2\beta^2 = 1$. The shear parameter for this solution can be very simply computed in the string frame

$$r(t_s) = \frac{3[H_s(t_s) - F_s(t_s)]}{[H_s(t_s) + 2F_s(t_s)]} \equiv \frac{3(\alpha - \beta)}{\alpha + 2\beta}. \tag{A.15}$$

Let us therefore do the same exercise in the Einstein frame. The scale factors and the cosmic time, transformed from the String to the Einstein frame are

$$a_E(t_E) = \left[-\frac{t_E}{t_1}\right]^{\frac{\alpha - 2\beta + 1}{3 - \alpha - 2\beta}}, \quad b_E(t_E) = \left[-\frac{t_E}{t_1}\right]^{\frac{1-\alpha}{3-\alpha-2\beta}}, \quad \left[-\frac{t_s}{t_1}\right] \simeq \left[-\frac{t_E}{t_1}\right]^{\frac{2}{3-\alpha-2\beta}}. \tag{A.16}$$

Therefore we get that

$$r(t_E) = \frac{3[H_E(t_E) - F_E(t_E)]}{[H_E(t_E) + 2F_E(t_E)]} \equiv \frac{6(\alpha - \beta)}{3 - (\alpha + 2\beta)}. \tag{A.17}$$

Now, by comparing Eqs. (A.15) and (A.17) it is clear that r_s and r_E go to zero in the same way and they are both proportional to $\alpha - \beta$. In this sense we can say that r is *approximately* invariant under conformal rescaling. For example by taking the anisotropic solution leading to flat axionic spectrum [5], namely $\alpha = -7/9$, $\beta = -4/9$ we get that $|r_s(t_s)| \sim 3/5$ whereas $|r_E(t_E)| \sim 3/7$. In all the cases discussed in this paper we explicitly checked that the shear parameters in one frame and in the other are of the same order of magnitude. So for practical purposes the shear parameters computed in the either in the String or in the Einstein frame have the same quantitative information *provided* the calculation is performed with respect to the correct cosmic time, which changes from frame to frame. More generally one would like to find a frame-independent measure of the degree of anisotropy. One could argue that these quantity is provided by the Weyl tensor and, in particular by ${C_{\mu\nu\alpha}}^{\beta}$. Indeed it is easy to show that ${C_{\mu\nu\alpha}}^{\beta} = {\mathcal{C}_{\mu\nu\alpha}}^{\beta}$ where the calligraphic style denotes, as usual in this Appendix, the quantities computed in the Einstein frame. Notice that the index position is crucial [20] since $C_{\mu\nu\alpha\beta} = e^{\phi}\mathcal{C}_{\mu\nu\alpha\beta}$. The only problem with ${C_{\mu\nu\alpha}}^{\beta}$ is that it is dimension-full. If we want to construct some dimension-less combination we run into the same ambiguity we just described since $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}/R_{\mu\nu\alpha\beta}R^{mu\nu\alpha\beta}$ transforms non trivially under conformal rescaling. So our conclusion is that the shear parameter is perhaps still the best quantity in order to characterize the degree of anisotropy. In our problem, moreover, we can show that the way the shear parameter goes to zero is a truly frame independent statement since, ultimately if $r_s(t_s) \propto |\alpha - \beta|$ we also have that $r_E(t_E) \propto |\alpha - \beta|$.

The second point we would like to stress concerns the covariant conservation of the (total) energy-momentum tensor in the Einstein frame. The Bianchi identities impose

$$\nabla_{\nu}\left[T_{\mu}^{\nu}(d) + e^{-\phi}T_{\mu}^{\nu}(m) + T_{\mu}^{\nu}(f)\right] = 0. \tag{A.18}$$

Now, it is easy to see that the 0 component of this equation, in the case of a magnetic field directed along the x axis, can be written as

$$(\partial_0 T_0^0(m) + 4FT_0^0(m))e^{-\phi} + \dot{\phi}\ddot{\phi} + \dot{\phi}^2(H + 2F) - T_0^0(m)e^{-\phi}\dot{\phi} + \dot{\rho} + (H + 2F)(\rho + p) = 0. \tag{A.19}$$

Using now Eq. (A.11) we see immediately that the mixed term $T_0^0(m)e^{-\phi}\dot{\phi}$ cancels. Thus, in order to satisfy the covariant conservation of the energy-momentum tensor we have to impose

$$\partial_0 T_0^0(m) + 4FT_0^0(m) = 0, \quad \dot{\rho} + (H + 2F)(\rho + p) = 0, \tag{A.20}$$

which implies that $T_0^0 = B^2/(2b^4)$ (where B is a constant) as correctly reported in Eqs. (A.8)–(A.12) from the very beginning as a consequence of the solutions of the equation for the field strength.

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